Assignment :- 2

• Attempt the following

1] If $z = -1 - i$ then $\arg(z)$ is equal to

a) $\frac{\pi}{4}$

2] Hyperbolic functions $\sinh x$ and $\cosh x$ are respectively

b) odd and even

3] Inverse hyperbolic function $\tanh^{-1} x$ is

a) $\frac{1}{2} \log \frac{1+x}{1-x}$

4] Modulus of complex number $z = x + iy$ is

c) $\sqrt{x^2+y^2}$

5] The differential equation $(x+y-2)dx + (x-y+4)dy = 0$ is of the form

a) exact

6] Real part of the complex number $z = e^{5 + i\frac{\pi}{2}}$ is

c) 0

7] Integrating factor of linear differential equation $\frac{dy}{dx} + Py = Q$ where P & Q are functions of y or constants

d) $e^{\int P dx}$

8] State Newton's law of cooling

Newton's law of cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the object's surroundings.

9] If the roots m_1 m_2 of Auxillary Equation $\phi(D)=0$ is

a) $C_1 e^{m_1 x} + C_2 e^{m_2 x}$

10] particular integral $\frac{1}{\phi(D)} e^{ax} V$ where v is any function of x nd $D = \frac{d}{dx}$ is

a) $\frac{1}{\phi(D+a)} \cancel{f(x)} e^{ax}$

11] particular integral $\frac{1}{\phi(D)} e^{ax} V$ where v is any function of x nd $D = \frac{d}{dx}$ is

c) $e^{ax} \frac{1}{\phi(D+a)}$

12] The solution of differential eqⁿ $\frac{d^2y}{dx^2} - y = 0$

Solve

1] for $x = \sqrt{3}$, find the value of $\tanh(\log x)$

→

$$\tanh(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}} = \frac{x - x^{-1}}{x + x^{-1}}$$

$$= \frac{x^2 - 1}{x^2 + 1} = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^2 + 1} = \frac{1}{2}$$

2] If the sum and product of two complex no. are real Show that those two numbers must be either used real or conjugate.

→ let,

$z_1 = x_1 + iy$ & $z_2 = x_2 + iy_2$

1) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$$\therefore \frac{(y_1 + y_2)}{(y_1 = -y_2)} = 0$$

2) $z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 \cdot x_2 + (x_1\, iy_2) + (iy_1 x_2) + (i^2 y_1 y_2)$

$z_1 z = (x_1, x) + i(x_1 y_2)(y_1 x_2) - y_1 y_2$

the product of complex number

$\therefore x_1 y_2 + y_1 x_2 = 0$

But $y_1 = -y_2$ Hence we get

$$x_1 y_2 - y_2 x_2 = 0$$
$$y_2(x_1 - x_2) = 0$$
$$y_2 = 0 \quad \text{or} \quad x_1 - x_2 = 0$$
$$x_1 = x_2$$

Case I :- If $y_2 = 0$

$$y_1 = 0$$
$$z_1 = x_1$$
$$\& \; z^2 = x^2$$

Case II :- If $x_1 = x_2$ & $y_2 \neq 0$

$$z_1 = x_1 + iy_1$$
$$z_1 = x_2 - iy_2$$

$$\therefore \overline{z_1} = x_2 - iy_2$$

$$z_1 = \overline{z_2}$$

3] Solve $(x+y-2)\,dx + (x-y+4)\,dy = 0$

→

Given :- $(x+y-2)\,dx + (x-y+4)\,dy = 0$

Compare given eq^n with

$$M\,dx + N\,dy = 0$$
$$M = x+y-2$$
$$N = x-y+4$$

$$M = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x+y-2)$$

$$\frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = 1$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Given differential eq^n is exact

Solution of exact differential equation

$$\int M\,dx + \int N\,dy = c$$

$$\int (x+y-2)\,dx + \int (x-y+4)\,dy = c$$

$$x^2 + xy - 2x + 0 - \frac{y^2}{2} + 4y = c$$

$$\frac{x^2}{2} + xy - 2x - \frac{y^2}{2} + 4y = c$$

4] If $\tan(A+iB) = x+iy$, prove that

1) $\tan 2A = \frac{2x}{1-x^2-y^2}$

2) $\tan 2B = \frac{2y}{1+x^2+y^2}$

→ Given

$$\tan(A+iB) = x+iy \quad \text{--- (i)}$$

$$\tan(A-iB) = x-iy \quad \text{--- (ii)}$$

let $(A+iB) = C$ —— ③

$(A-iB) = D$ —— ④

Adding ③ & ④

$$2A = C+D$$

Apply tan on both sides

$$\tan 2A = \tan(C+D)$$

$$= \frac{\tan C + \tan D}{1 - \tan C \cdot \tan D}$$

$$= \frac{\tan A+iB + \tan A-iB}{1-\tan A+iB \cdot \tan A-iB}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2A = \frac{2x}{1-x^2-y^2}$$

Subtracting ③ & ④

$$2iB = C-D$$

Apply tan on both sides

$$\tan(2iB) = \tan(C-D)$$

$$i \tanh 2B = \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$$

$$i \tan h 2B = \frac{2iy}{1+x^2+y^2}$$

$$\therefore \tan h 2B = \frac{2y}{1+x^2+y^2}$$

5] Find complementary function $(D^4+2D^2+1)\,y=0$

→ Given:-

$$D^4+2D^2+D\,y=0$$

Auxiliary eq^n is

$$m^4+2m^2+1=0$$

let

$$m^2=p$$

$$p^2+2p+1=0$$

$$p=-1,\ -1$$

$$m^2=-1,\quad m^2=-1$$

$$m=\pm\sqrt{-1},\ \pm\sqrt{-1}$$

This one repeated complex roots

$$y_c = e^{\alpha}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x]$$

Here $\alpha=0$ $\beta=1$

$$y_c = e^{x}[(c_1+c_2x)\cos 1x + (c_3+c_4x)\sin 1x]$$

$y_c = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

6] Solve $(D^6 - D^4)y = x^2$

→ Given :-

$D^6 - D^4 y = x^2$

A.E $m^6 - m^4 = 0$

$m^4(m^2 - 1) = 0$

$m^4(m+1)(m-1) = 0$

$m = 0, 0, 0, 0, 1, -1$

$y_c = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{0x} + (c_5 e^{1x} + c_6 e^{-1x})$

$y_c = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x}$

To Find $y_p = \frac{1}{D^6 - D^4} \cdot x^2$

$y_p = \frac{1}{D^4(D^2 1)} \cdot x^2$

$y_p = \frac{1}{D^4(1 - D^2)} x^2$

Using $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots x^n$

Here $x = D^2$

$$yp = \frac{-1}{D^4}\left(\frac{1}{1-D^2}\right)x^2$$

$$yp = \frac{1}{D^4}\left[1 + D^2 + D^4 + D^6 + \cdots\right]x^2$$

$$yp = \frac{-1}{D^4}\left[x^2 + 2 + 0 + D \cdots\right]$$

$$yp = \frac{-1}{D^3}\left[\frac{x^3}{3} + 2x + c\right]$$

$$= \frac{-1}{D^2}\left[\frac{x^4}{12} + \frac{\cancel{2}x^2}{\cancel{2}} + c_1 x + c_2\right]$$

$$yp = \frac{-1}{D}\left[\frac{x^5}{60} + \frac{x^3}{3} + \frac{c_1 x^2}{2} + c_2 x + c_3\right]$$

$$yp = -1\left[\frac{x}{360} + \frac{x^4}{12} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4\right]$$

$$y = y_c + yp$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x}$$

$$\left[\frac{x^6}{360} + \frac{x^4}{12} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4\right]$$

Q.3. Solve

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1] Solve $(D^2+2D+1)y = 4\sin 2x$

→ Given :-

$$(D^2+2D+1)y = 4\sin 2x$$

$\therefore$ AF $\Rightarrow$ $m^2+2m+1=0$

$m = -1, -1 \longrightarrow$ Repeated Real Roots

$y_c = (c_1 + c_2 x) \cdot e^{-1x}$

To find :- y_p

$$y_p = \frac{1}{D^2+2D+1} 4\sin 2x$$

put $D^2 = -a^2 = (-2^2) = -4$

$$y_p = \frac{1}{-4+2D+1} \cdot 4\sin 2x$$

$$y_p = \frac{1}{2D-3} 4\sin 2x$$

Using Rationalisation of binominator

$$y_p = \frac{2D+3}{(2D-3)(2D+3)} 4\sin 2x$$

$$y_p = \frac{2D+3}{4D^2-9} \cdot 4\sin 2x$$

put $D^2 = (-2^2) = -4$

$$= \frac{2D+3}{4(-4)-9} 4\sin 2x$$

$$= \frac{2D+3}{-16-9} 4\sin 2x$$

$$y_p = \frac{2D+3}{-25} 4\sin 2x$$

$$= \frac{2D(4\sin 2x)+3(4\sin 2x)}{-25}$$

$$= \frac{8\cos 2x \times 2 + 12\sin 2x}{-25}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{16\cos 2x + 12\sin 2x}{-25}$$

2] Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

→

Given:-

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

Divide by $\cos^2 x$

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$$\frac{\cos^2 x}{\cos^2 x}\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x$$

compare with $\frac{dy}{dx} + Py = Q$, we get

$$\frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x$$

$P = \sec^2 x$, $Q = \tan x \cdot \sec^2 x$

$T \cdot F \; e^{\int P dx}$

$= e^{\int \sec^2 x \, dx}$

$T \cdot F = e^{\tan x}$

General solution is

$y \cdot TF = \int Q \cdot IF \, dx$

$y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \, (e^{\tan x}) \, dx$

put $\tan x = t$

Diff w.r.t. x

$\sec^2 x = \frac{dt}{dx}$

$\sec^2 x \, dx = dt$

$y \cdot e^{\tan x} = \int t \cdot e^t \, dt$

using integration $\int u \cdot v \, dx$

$\int u \cdot v \, dx = u \int v \, dx - \frac{du}{dx} \iint v \, dx + \frac{d^2 y}{dx^2} \iiint y \, dx$

$y \cdot e^{\tan x} = t \int e^t \, dt - \frac{d(t)}{dt} \iint e^t \cdot dt + \frac{d^2}{dt^2}(t) \iiint e^t \, dt$

$y \cdot e^{\tan x} = t \cdot e^t - 1 \cdot e^t + 0 + C$

put $t = \tan x$

$y \cdot e^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + C$

3] Solve $x^3 - 1 = 0$

→

$x^3 - 1 = \cos 0 + i \sin 0 = \cos(2n\pi) + i \sin(2n\pi)$

$x = ((\cos(2n\pi) + i \sin(2n\pi))^{1/3}$

$= \cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3}$

putting $n = 0, 1, 2,$

$x_0 = \cos 0 + i \sin 0 = 1$

$$x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$$

$$x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

$\omega = (-1/2 + i\sqrt{3}/2)$ and

$$\omega^2 = (-1/2 - i\sqrt{3}/2).$$

4] Solve the Equation $x^6 - i = 0$

→ Given:-

$$x^6 - i = 0$$

$$x^6 = i \qquad \therefore x = (i)^{\frac{1}{6}} = \theta ----- ①$$

Now,

$$I = 0 + i = r[\cos\theta + i\sin\theta]$$

$$Z = \theta = \tan^{-1}\left|\frac{y}{x}\right|$$

Where $r = \sqrt{0+1} = 1$

and $\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

using standard trigonometric value

$$--- \left[\tan\left(\frac{\pi}{2}\right) = \infty \Longrightarrow \tan^{-1}(\infty) = \left(\frac{\pi}{2}\right)\right]$$

$$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

$i = \cos\left(2\pi m + \frac{\pi}{2}\right) + i\sin\left(2\pi m + \frac{\pi}{2}\right)$ --- General polar form

$$i = \cos\left(\frac{4\pi m + \pi}{2}\right) + i\sin\left(\frac{4\pi m + \pi}{2}\right)$$

Taking 6^{th} roots of both sides

$$(i)^{\frac{1}{6}} = \left[\cos\left(\frac{4\pi m + \pi}{2}\right) + i\sin\left(\frac{4\pi m + \pi}{2}\right)\right]^{\frac{1}{6}}$$

Using De-moivre's theorem ----

$$[(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

$$(i)^{\frac{1}{6}} = \left[\cos\left(\frac{1}{6} \times \frac{4\pi m + \pi}{2}\right) + i\sin\left(\frac{1}{6} \times \frac{4\pi m + \pi}{2}\right)\right]$$

$$(i)^{\frac{1}{6}} = \cos\left(\frac{4\pi m + \pi}{12}\right) + i\sin\left(\frac{4\pi m + \pi}{12}\right)$$

Where $m = 0, 1, 2, 3, 4, 5$

Substitute this value in eq^n (1)

$$x = \cos\left(\frac{4m+1}{12}\right)\pi + i\sin\left(\frac{4m+1}{12}\right)\pi$$

$m = 0, 1, 2, 3, 4, 5$

For

$m = 0 \qquad x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$

$m=1 \qquad x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$m=2 \qquad x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$

$m=3 \qquad x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$m=4 \qquad x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$m=5 \qquad x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}$

These are roots of $x^6 - i = 0$

5) A metal ball is heated to a temperature of 100°C and at time $t=0$ it is placed in water which is maintained at 40°C. If the temperature of the ball is reduced to 60°C in 4 minutes. Find the temp at which the temperature of the ball is 50°C.

→

Let the temperature of the ball the T°C at time t min. Then the differential eqⁿ is :

$$\frac{dT}{dt} = -K(T-40) \quad \text{—— (1)}$$

Integration gives, $-kt = \log(T-40) + \log C$ —— (2)

At $t=0$, $T=100$

This gives $\log C = -\log 60$

hence eqⁿ (2) becomes,

$$-kt = \log\frac{T-40}{60} \quad \text{—— (3)}$$

$$-4k = \log \frac{1}{3}$$

$$k = \frac{1}{4} \log 3$$

Hence, eq^n 3 gives

$$-\frac{t}{4} \log 3 = \log \frac{T-40}{60}$$

when $T = 50$, we obtain

$$t = \frac{4 \log 6}{\log 3} = 6.5 \text{ minutes}$$

6) Solve $(x^2 + y^2)\,dx - xy\,dy = 0$

→ Given :-

$$(x^2 + y^2)\,dx - xy\,dy = 0$$

$$\left(\frac{x^2 + y^2}{xy}\right) = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{(x^2 + y^2)}{xy} \quad \text{——— (1)}$$

Degree of each term is same

∴ This is homogeneous diff^n eq^n

put $\underline{\underline{y = ux}}$

Diff w.r.t. x

$$\frac{dy}{dx} = u \cdot \frac{d}{dx} x + x \frac{du}{dx}$$

$$\frac{dy}{dx} = u \cdot 1 + x \frac{du}{dx}$$

Eq^n ① becomes

$$u + x\frac{du}{dx} = \frac{x^2 + u^2 x^2}{u(ux)}$$

$$u + x\frac{du}{dx} = \frac{\cancel{x^2}(1+u^2)}{\cancel{x^2}\, u}$$

$$u + x\frac{du}{dx} = \frac{1+u^2}{u}$$

$$x\frac{du}{dx} = \frac{1+u^2}{u} - u$$

$$x\frac{du}{dx} = \frac{1+u^2-u^2}{u}$$

$$x\frac{du}{dx} = \frac{1}{u}$$

$$x\,du = \frac{1}{u}\,dx$$

$u\,du = \frac{1}{x}\,dx$ —— variable seperable form

Integrating both sides

$$\int u \cdot du = \int \frac{1}{x}\,dx$$

$$\frac{u^2}{2} = \log x + c$$

put $u = \frac{y}{x}$

$$\frac{y^2}{2x^2} = \log x + c$$